

Aspects of the stochastic Burgers equation and their connection with turbulence.

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Summary.

We present results for the 1 dimensional stochastically forced Burgers equation when the spatial range of the forcing varies. As the range of forcing moves from small scales to large scales, the system goes from a chaotic, structureless state to a structured state dominated by shocks. This transition takes place through an intermediate region where the system exhibits rich multifractal behavior. This is mainly the region of interest to us. We only mention in passing the hydrodynamic limit of forcing confined to large scales, where much work has taken place since that of Polyakov[1].

In order to make the general framework clear, we give an introduction to aspects of isotropic, homogeneous turbulence, a description of Kolmogorov scaling, and, with the help of a simple model, an introduction to the language of multifractality which is used to discuss intermittency corrections to scaling.

We continue with a general discussion of the Burgers equation and forcing, and some aspects of three dimensional turbulence where - because of the mathematical analogy between equations derived from the Navier-Stokes and Burgers equations - one can gain insight from the study of the simpler stochastic Burgers equation. These aspects concern the connection of dissipation rate intermittency exponents with those characterizing the structure functions of the velocity field, and the dynamical behavior, characterized by different time constants, of velocity structure functions. We also show how the exponents characterizing the multifractal behavior of velocity struc-

ture functions in the above mentioned transition region can effectively be calculated in the case of the stochastic Burgers equation.

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I. Introduction.

We study some aspects of statistically stationary, homogeneous and isotropic fully developed turbulence. This is the typical framework in which such studies are done. The quantities of interest are the equal time spatial correlations of the velocity field $\vec{u}(\vec{x}, t)$, the so-called structure functions. The longitudinal structure functions, which are the ones usually discussed, are defined by

$$S_p(r) = \langle [(\vec{u}(\vec{x} + \vec{r}, t) - \vec{u}(\vec{x}, t)) \cdot \vec{n}]^p \rangle \quad (1)$$

where \vec{n} is the unit vector in the direction \vec{r} . Some components of the velocity field difference can be projected onto the direction transverse to \vec{n} , and thus there are other correlations of $p - th$ order, which involve longitudinal and an (even) number of transverse projections.

The velocity satisfies the incompressible Navier-Stokes equation

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} p + \nu \Delta \vec{u} + \vec{f} \quad (2)$$

with

$$\vec{\nabla} \cdot \vec{u} = 0 \quad (3)$$

Here p is the pressure divided by the constant mass density, ν the kinematic viscosity. We have added $\vec{f} = \vec{f}(\vec{x}, t)$, an external stochastic force which acts on large scales, and maintains a turbulent steady state. The average in (1) then includes as well an average over time.

In the usual picture of turbulence (see I.1.), when the distance $r = |\vec{r}|$ in (1) is small compared to large scales L of the order of the system size, and large compared to the scales where dissipation takes place, the structure functions are expected to behave as

$$S_p(r) \sim (r/L)^{\zeta_p} \quad (4)$$

An important aspect of solving the problem of statistical isotropic, homogeneous turbulence is deriving the values of the exponents ζ_p in (4) from the Navier-Stokes equation. This has not been done except for ζ_3 , the value of which is fixed by the Von Karman-Howarth relation[2]. It turns out however that the experimentally measured ζ_p 's (up to $p = 10$ or so) are not too different from their scaling values as they arise in the picture of fully developed turbulence proposed by Kolmogorov. This is the reason a large number of phenomenological models exist, which by breaking scale invariance slightly, give improved fits to the data. The usual language in which to express deviations from scaling is that of multiscaling or multifractality.

We will therefore discuss first in this introductory section Kolmogorov scaling, then a simple model, which allows one to introduce non-scaling elements, and provides a simple introduction to the language of multiscaling which we present next. A general reference for these subjects is the book of Frisch[3].

In the second section we discuss the stochastic Burgers equation, its shock structure and the associated extreme multifractality, and its behavior when the spatial range of the random forcing varies from small to large scales. In section III we take up the point about statistical aspects of the stochastic Burgers equation and their connection with three dimensional, forced, isotropic and homogeneous turbulence. First we show how the problem of multifractality can be solved for the stochastic Burgers equation. Then we discuss the relation between intermittency in the energy dissipation to intermittency in the velocity field, and end up by making a number of observations concerning the dynamical behavior of structure functions. General remarks about intermittency in fully developed turbulence and for the stochastic Burgers equation are made in section IV.

This report is based on a number of results or points made in references[4, 5, 6, 7, 8].

I.1. Kolmogorov scaling.

The picture is that of an energy cascade from the large scale L where the energy is put into the system, to the dissipation scale δ where it is dissipated. On intermediate scales $\delta \ll r \ll L$, which make up the so-called inertial range, the only quantity which matters is ϵ , the mean energy dissipation rate per unit mass, considered to be independent of scale. ϵ has the dimension of velocity squared divided by time, or velocity cubed divided by distance.

The dissipation scale δ can only depend on ν and ϵ , and thus for dimensional reasons $\delta \sim (\nu^3/\epsilon)^{\frac{1}{4}} \sim (1/Re)^{\frac{3}{4}}L$, where after replacing ϵ in terms of a characteristic velocity U and the large scale L , we are able to introduce the Reynolds number $Re = UL/\nu$. In the limit of small viscosity or large Reynolds number there is thus a definite separation of scales between δ and L .

In the inertial region, dimensions are determined by ϵ alone, and therefore one predicts on dimensional grounds, that $S_p(r)$ which has the dimension of velocity to the $p - th$ power behaves as

$$S_p(r) \sim \epsilon^{\frac{p}{3}} r^{\frac{p}{3}} \quad (5)$$

This is Kolomogorov scaling. The scaling values of the exponents in (4) are then

$$\zeta_p = p/3 \quad (6)$$

This gives $\zeta_2 = 2/3$, which by Fourier transform is equivalent to the experimentally observed $-5/3$ behavior of the energy spectrum, namely $E(k) = k^2 < \vec{u}(\vec{k}) \cdot \vec{u}(-\vec{k}) > \sim \epsilon^{2/3} k^{-5/3}$. One also obtains $\zeta_3 = 1$, which is the value fixed by the Von Karman-Howarth relation. The other general result[3] is that ζ_p is a convex function of p . Measurements of the structure functions show[9] however that Kolmogorov scaling does not hold: the measured ζ_p 's for $p > 3$ lie below the scaling values. For instance $\zeta_6 = 1.80 \pm 0.05$ rather than the scaling value of 2, obtained from (6). This effect is called intermittency or multifractality, and can be related heuristically to the non-space filling property of the eddies which make up the energy cascade, and therefore to their fractal dimension. A simple model will serve to illustrate these points.

1.2. A simple model.

Among models which describe the energy cascade, the so-called β -model [10] is instructive. Imagine, as the energy cascades down to smaller scales from the large scale L , that at scales $r = \alpha^n L$ in the inertial range, the eddies at this scale, which themselves have a typical size of r , occupy only a fraction β of the available space, such that $p_r = \beta^n$, where p_r can be interpreted as the probability of finding an eddy of size r at scale r . Eliminating the "generation" number n between the expressions for r and p_r , one finds

$$p_r = (r/L)^{3-D} \quad (7)$$

where $3 - D = \ln \beta / \ln \alpha$. If the eddies are space filling, then $\beta = 1$, and therefore $D = 3$. The value of 3 corresponds to the fact that we pretend our discussion is for eddies in 3 dimensions. The argument itself is clearly independent of space dimension. One now interprets D as the fractal dimension of the space on which the eddies exist, assuming that D is smaller than 3.

What are the structure functions in this model?

The typical energy of an eddy of size r is $E_r \sim \delta v_r^2 p_r$, and therefore the average energy dissipation rate (per unit mass) at scale r , with a typical time scale $t_r = r/\delta v_r$, is

$$\epsilon_r \sim \frac{\delta v_r^3}{L} (r/L)^{3-D-1} \quad (8)$$

Here δv_r is the velocity variation across the eddy. The value of ϵ_r is independent of r if homogeneity holds (existence of an inertial scale), and therefore one has for the velocity

$$\delta v_r \sim (\epsilon L)^{1/3} (r/L)^{\frac{1}{3} - (3-D)/3} \quad (9)$$

from which follows for the structure function

$$S_p(r) = \langle \delta v_r^p \rangle = \delta v_r^p p_r \sim (\epsilon L)^{p/3} (r/L)^{p/3 + (3-D)(1-p/3)} \quad (10)$$

One thus finds for the exponents ζ_p of the structure functions, a convex function of p , namely

$$\zeta_p = p/3 + (3-D)(1-p/3) \quad (11)$$

which satisfies the condition (Von Karman-Howarth relation) $\zeta_3 = 1$. The scaling violating part in ζ_p is given by $(3-D)(1-p/3)$. For instance $\zeta_6 = 2 - (3-D)$, which, by comparison with the experimental result $\zeta_6 = 2 - 0.2$, leads to a fractal dimension $D = 2.8$. Note that the velocity variation at r ($\delta v_r \sim r^h$) is itself characterized by an exponent $h = 1/3 - (3-D)/3$. For $D = 3$, when the eddies fill all space at any inertial scale, one has the scaling (Kolmogorov) result $h = 1/3$ and $\zeta_p = p/3$.

In the simple model we have considered, the structure functions and the variations of the velocity field are characterized by a single h and D . However here, as opposed to the Kolmogorov scaling behavior, the eddies are not space filling, but are characterized by a fractal dimension D .

Simple fractal models such as the one we have described are not believed to give the whole picture required to describe fully developed turbulence. Experimental data suggest that ζ_p depends non linearly on p in contrast to equation (11). It is believed[3] that one needs to consider a more general picture, with a range of possible h 's and of corresponding fractal dimensions $D(h)$ (see section IV.). This picture, or the language in which it is formulated, is that of multifractality, which we discuss next.

I.3. The language of multifractality.

Assume now that h can take values in an interval (h_{min}, h_{max}) , and that to each h there corresponds a set in three dimensional space of fractal dimension $D(h)$, in such a way that across any distance r (r belongs to the inertial range) in the vicinity of that set, one has

$$\delta v_r \sim (r/L)^h \quad (12)$$

and

$$p_r \sim (r/L)^{3-D(h)} \quad (13)$$

where p_r is the probability for being within a distance of the set of fractal dimension $D(h)$, and δv_r is the velocity variation. As a consequence one has the following expression for $S_p(r)$ for a given set with scaling dimension h

$$S_p(r) \sim \langle \delta v_r^p \rangle \sim (r/L)^{ph+3-D(h)} \quad (14)$$

All h can contribute to the right-hand side, but since $r/L \ll 1$, the dominant exponent ζ_p is given by

$$\zeta_p = \min_h (ph + 3 - D(h)) \quad (15)$$

This exponent ζ_p is the dominant one in the expression of the structure factors (cf. equation (4)).

Remarks:

- the scaling result corresponds to $h = 1/3$ and $D(1/3) = 3$.
- the argument is the same in 1 or 2 dimensions with the replacement of the number 3 in $3 - D(h)$ by respectively 1 and 2.
- the quantity $3 - D(h)$ is positive or zero, since $D(h)$ cannot exceed the dimension of the embedding space. It is generally assumed that $h_{min} \geq 0$. In the case of the Burgers equation where exponents can be calculated, we find (cf. section III.1.) that the h 's corresponding to higher order structure functions reach the value 0 when the stochastic forcing has moved to sufficiently large scales, and stay at the value 0 when the scale of the forcing increases further.

II. The stochastic Burgers equation.

This is a 1 dimensional version of the Navier-Stokes equation, a version without incompressibility and pressure, which describes the evolution of the compressible field $u(x, t)$, by

$$\partial_t u + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f \quad (16)$$

where $f = f(x, t)$ is a stochastic forcing.

We will discuss later the forcing and its influence on the dynamics of the field. For the moment, we will ignore it, and summarize some results concerning the plain Burgers equation[11].

II.1. Shock structure and extreme multifractality.

If one starts from an initial sinusoidal velocity profile of large wavelength, then under the influence of the nonlinear term in the equation, the sinusoid will for sufficiently small viscosity, steepen into a series of shocks. After some time the shocks will fade away, their energy being dissipated by the viscous term. This viscous term plays a role mainly at the position of the shocks, where it is counterbalanced by the nonlinear term. The equality of these two terms leads to

$$\nu = \Delta u \cdot \delta \quad (17)$$

where Δu is the velocity jump across the shock, and δ is the shock thickness. There are thus two scales here: a large scale L corresponding to some average distance between shocks, and a dissipation scale $\delta \sim \nu$, very much smaller than L when ν goes to zero. Distances away from both extremes make up the inertial range.

In terms of multifractal language, the Burgers equation (one averages, in the limit $\nu \rightarrow 0$, over an ensemble of initial states, or considers stochastic forcing on large scales) shows extreme multifractality, a situation called bifractality in the literature[3]. The behavior of u is essentially linear between shocks ($u \sim x$), and thus here $h = 1, D(1) = 1$. At the shocks themselves $h = 0, D(0) = 0$, since the shocks are discontinuities of the velocity field occurring at a point (in the $\nu \rightarrow 0$ limit). The velocity variation across the shock is independent of distance, and the probability of being within a distance r is linear in r (cf. equations (12) and (13) for the case of 1 dimension).

There are thus two possible values for the exponent $ph + 1 - D(h)$ (cf. section I.3.), namely p or 1, and therefore the dominant exponent ζ_p (equation (15)) characterizing the behavior of the structure functions in the inertial scale, is such that

$$\zeta_p = 1, \quad p \geq 1 \quad (18)$$

This is an extreme case of multifractality (all exponents have the same value for integer p greater than 1), very much different from the case of three dimensional homogeneous, isotropic turbulence where the experimentally determined exponents remain relatively close to the scaling ones, which increase linearly with p (see equation (6)).

However - as we have discovered - there is a whole range of multifractal behavior as the spatial extent of the stochastic force in the Burgers equation varies, and the situation is much more interesting.

II.2. Stochastic forcing.

For the stochastic forcing in (16) we take a Gaussian, such that in k space

$$\begin{aligned} \langle f(k, t) \rangle &= 0 \\ \langle f(k, t) f(k', t') \rangle &= 2D_0 |k|^\beta \delta_{k, -k'} \delta(t - t') \end{aligned} \quad (19)$$

The exponent β determines over which scales the forcing acts. For $\beta > 0$ it acts effectively on small scales, whereas as β becomes negative, larger and larger scales matter. The limit relevant to forcing in three dimensional turbulence is that of large scales, of the order of the system size L .

The range of values of β goes from $\beta = 2$, which corresponds to thermal noise, to $\beta = -3/2$. For values smaller than the latter, the statistics of the velocity field is independent of β , unchanged from its behavior at $\beta = -3/2$. At $\beta = -3/2$ the system behaves as the steady state of the plain Burgers equation: it exhibits the extreme multifractal behavior discussed in II.1., characteristic of a shock dominated velocity field. For $\beta > 0$ however, the presence of noise on small scales prevents the shocks from developing, and therefore the behavior appears chaotic, i.e. random and structureless. Thus as β moves from positive to large negative values, the velocity field goes from a chaotic to a shock dominated state, through an intermediate region[12] ($-3/2 < \beta < 0$), where for $-1 < \beta < 0$ it displays complex dynamics of appearing, interacting and disappearing shocks. This region is one of rich multifractal behavior, and is the principal object of our study. It is through this region that one approaches the hydrodynamic limit of large scale forcing from a purely chaotic state.

To be complete, we mention that for positive values of β one can use a renormalization group approach. As soon as β becomes negative, all sorts of non-linear terms become important in the equations, and the perturbative renormalization group approach breaks down. This approach has been usually applied[13] to the equivalent KPZ (Kardar-Parisi-Zhang) equation for fluctuations of an interface height $h(x, t)$, related to u by $u = \partial_x h$. With a noise of the form considered, the renormalization group has also been applied to the Navier-Stokes equation[14].

For β positive, close to zero, the scaling analysis leads to the following result for the exponents z and ζ_2 , which appear in the scaling form assumed for $S_2(r, \tau) = \langle (u(x + r, t + \tau) - u(x, t))^2 \rangle$, namely $S_2(r, \tau) = r^{\zeta_2} g(\tau/r^z)$:

$$z + \zeta_2/2 = 1 \quad (20)$$

and

$$\zeta_2 - z = -1 - \beta \quad (21)$$

The first relation is a consequence of Galilean invariance, the second of the fact that the coefficient D_0 of noise fluctuations is not rescaled because of the non-analytic form of the noise. One obtains from (20) and (21) that $\zeta_2 = -2\beta/3$ and $z = 1 + \beta/3$.

We will from now on consider the region of negative β , which is so to speak the gateway to hydrodynamic behavior.

III. Three dimensional turbulence and the stochastic Burgers equation.

We believe that because of the mathematical similarity of the Navier-Stokes equation with forcing, and the stochastic Burgers equation, the latter can be used as a key to the understanding of a number of issues in the statistical behavior of isotropic, homogeneous turbulence. In the work we have been doing[4, 5, 6, 7, 8], we highlight this similarity on a number of occasions, in different situations. To give a simple example here, we compare the Von Karman-Howarth relation for S_3 for both equations.

For the Navier-Stokes equation with forcing \vec{f} , this relation takes the following form for the (equal time) 3rd order structure function $S_{3j} = \langle (\vec{u}_1 - \vec{u}_2)^2 (u_{1j} - u_{2j}) \rangle$, where "1" refers to the point $\vec{x} + \vec{r}$, "2" to the point \vec{x} , and "j" denotes the j-th component of \vec{u}

$$\frac{1}{2} \partial_{r_j} S_{3j}(r) = \nu \Delta S_2(r) - 2 \langle \epsilon \rangle + \langle (\vec{u}_1 - \vec{u}_2) \cdot (\vec{f}_1 - \vec{f}_2) \rangle \quad (22)$$

where $S_2(r) = \langle (\vec{u}_1 - \vec{u}_2)^2 \rangle$, while for the stochastic Burgers equation, where $S_3(r) = \langle (u_1 - u_2)^3 \rangle$, it reads

$$\frac{1}{6} dS_3(r)/dr = \nu d^2 S_2/dr^2 - 2 \langle \epsilon \rangle + \langle (u_1 - u_2)(f_1 - f_2) \rangle \quad (23)$$

The structural similarity of the two equations is clear.

One can derive the above two Von Karman-Howarth relations in a straightforward way from the space and time dependent S_2 , using the homogeneity in time of expectation values. More precisely, one writes that $\partial S_2(r, \tau)/\partial t_1 + \partial S_2(r, \tau)/\partial t_2 = 0$, where $r = x_1 - x_2, \tau = t_1 - t_2$. This derivation highlights the fact, which we have several times pointed out in our work, that it is often useful for deriving equal time correlations to pass through time dependent calculations. Many identities can be obtained this way.

The two equations (22) and (23) are very similar. The 3 dimensional result contains Kolmogorov's "4/5th" law for the longitudinal structure function. In both cases $\langle \epsilon \rangle$ represents the energy dissipation rate. Since r belongs to the inertial scale the term multiplied by ν is negligible in both equations in the zero viscosity limit. The noise dependent term can be evaluated in the equal time limit with the help of the Novikov-Donsker formalism[15]. When the noise is cut-off at large scales (the hydrodynamic limit) this term leads to a subdominant correction of order $(r/L)^2$. We will discuss later, for the stochastic Burgers equation, the general case when the noise ranges over small scales as well.

Though this comparison of the Von Karman-Howarth relations is based on a simple case, we have found that the same similarity term by term, with an obvious display of the 3 dimensional space indices, holds for any other equation we have derived involving velocity or dissipation rate correlations, with the exclusion of course of terms involving pressure.

We will discuss in the following three main points:

(i) first, we are going to face for the stochastic Burgers equation the problem of turbulence, namely calculate, for small p , in the multifractal region $(-1 < \beta < 0)$ the exponents ζ_p characterizing the statistical behavior of velocity structure functions,

(ii) second, we are going to give the general equation satisfied by the equal time correlation of the dissipation rate, and connect its intermittent behavior, which exhibits a hierarchy of exponents, to the intermittent behavior of the velocity structure functions,

(iii) third, we investigate the dynamics of the second order structure function, and show how - even in the absence of any average flow - S_2 satisfies a wave equation with characteristic velocity $\sqrt{\langle u^2 \rangle}$. These dynamic considerations enable us to disentangle, in our Eulerian framework, the intrinsic dynamical and the kinetic, ballistic characteristic times which describe the time evolution of flow structures.

III.1. Multifractal exponents.

We are interested in the region where $-1 < \beta < 0$. Here also exists the possibility of scaling behavior, in the same way as there is Kolmogorov scaling for three dimensional turbulence, where the dimension of $\langle \epsilon \rangle$ or equivalently D_0 , determines the dependence on distance of the S_p 's in the inertial range. One thus has

$$S_p(r) \sim (D_0/L)^{p/3} r^{-p\beta/3} \quad (24)$$

which corresponds to $\zeta_p = -p\beta/3$ and $h = -\beta/3$. This is the value of h in the scaling regime. (Notice that at $\beta = -1$ the exponents are the same[16] as those of Kolmogorov scaling, equation (6).)

This scaling regime is however dominant only in the region of β negative close to zero, and gives way to multifractal behavior as β goes towards -1 . We are going to study this behavior directly on equations for the structure functions derived from the stochastic Burgers equation. We proceed systematically discussing first S_2 and S_3 , and then S_4, S_5 and general S_p .

(i) S_2 and S_3 .

One cannot derive directly from the stochastic Burgers (or from the Navier-Stokes equation in three dimensions) a closed equation for the equal time structure function S_2 . We therefore check numerically that $S_2(r)$ behaves in the following way

$$S_2(r) \sim (r/L)^{-2\beta/3} \quad (25)$$

for all $-3/2 < \beta < 0$. Precise numerical results, and therefore a precise value of the exponent, can be obtained from evaluating the energy spectrum ($E(k) \sim |k|^{-1+2\beta/3}$), related to S_2 by Fourier transform, rather than from S_2 itself (Figure 1). $S_2(r)$ thus scales, in the sense that $\zeta_2 = -2\beta/3$ has its scaling value (cf. equation (24)).

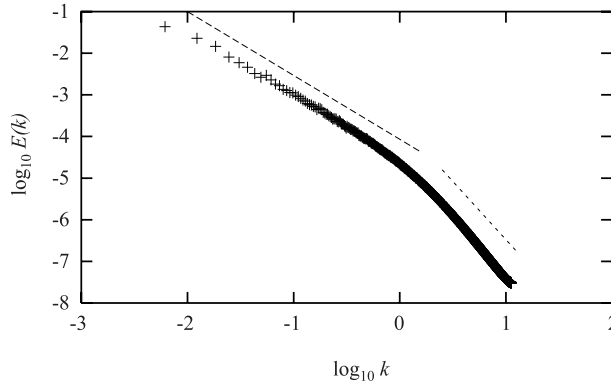


Figure 1: Graph of $\log E(k)$ as a function of $\log k$, where the energy spectrum is $E(k) \sim |k|^{-1+2\beta/3}$, for $\beta = -0.8$. The straight line for small k , drawn for comparison, has a slope of -1.53 , which is the value of $-1 + 2\beta/3$ at the given β .

As to $S_3(r)$, it is determined from the Von Karman-Howarth relation, equation (23). In this equation the noise term takes in the equal time limit

(Novikov-Donsker formalism[15]) the form

$$\langle (u_1 - u_2)(f_1 - f_2) \rangle = 2(1/L^2) \sum_k D_0 |k|^\beta (1 - \cos kr) \quad (26)$$

The term proportional to "1" in $(1 - \cos kr)$ cancels the -2ϵ in equation (23) because $(1/L^2) \sum_k D_0 |k|^\beta$ is the total rate of energy input. One thus obtains from equation (23) (in the $\nu \rightarrow 0$ limit)

$$\frac{1}{6} dS_3/dr = -2(1/L^2) \sum_k D_0 |k|^\beta \cos kr \quad (27)$$

The " $\cos kr$ " term leads by rescaling to the following result

$$S_3(r) \sim r^{-\beta} \quad (28)$$

for $-1 < \beta < 0$, in the case where the noise does not have a cut-off at scales of order L . (At $\beta = -1$ there is an additional logarithm, $S_3 \sim r \log r$.)

The exponents characterizing the inertial range behavior of S_2 and S_3 have therefore their scaling values throughout the domain $-1 < \beta < 0$. For S_2 the result is based on simulations, for S_3 the expression of the exponent is obtained from the Von Karman-Howarth relation.

(ii) S_4, S_5 **and general** S_p .

For $p \geq 4$ scaling no longer holds through the entire $-1 < \beta < 0$ range. The following are the equations we obtain from the stochastic Burgers equation after isolating the terms which in the inertial range go to zero when the viscosity does, and simplifying the noise terms

$$\frac{1}{6} dS_4(r)/dr = \frac{2}{3} \nu d^2 S_3/dr^2 - 2 \langle (\epsilon_1 + \epsilon_2)(u_1 - u_2) \rangle \quad (29)$$

$$\begin{aligned} \frac{1}{40} dS_5(r)/dr &= \frac{1}{12} \nu d^2 S_4/dr^2 - \frac{1}{2L^2} \sum_k D_0 |k|^\beta \cos(kr) \langle (u_1 - u_1)^2 \rangle \\ &\quad - \frac{1}{2} [\langle (\epsilon_1 + \epsilon_2)(u_1 - u_2)^2 \rangle \\ &\quad - \langle (\epsilon_1 + \epsilon_2) \rangle \langle (u_1 - u_2)^2 \rangle] \end{aligned} \quad (30)$$

$$\begin{aligned} dS_p(r)/dr &\sim \frac{1}{2L^2} \sum_k D_0 |k|^\beta \cos(kr) \langle (u_1 - u_2)^{p-3} \rangle \\ &\quad + \dots \langle (\epsilon_1 + \epsilon_2)(u_1 - u_2)^{p-3} \rangle \end{aligned} \quad (31)$$

The right-hand sides of equations (29) and (30) contain terms (not written for equation (31)) which go to zero in the small viscosity limit, a noise dependent term and a dissipation rate dependent term. The noise term has the general form

$$\sum_k D_0 |k|^\beta \cos(kr) < (u_1 - u_2)^{p-3} > \sim \frac{dS_3(r)}{dr} S_{p-3}(r) \quad (32)$$

since $dS_3(r)/dr \sim \sum_k D_0 |k|^\beta \cos kr$ (cf. equation (27))

Therefore scaling behavior in S_p is present, whether dominant or subdominant, whenever there is scaling behavior in S_{p-3} . Thus the presence of a scaling term in S_2, S_3 and S_4 guarantees the presence of one in any S_p for $p \geq 4$. We have already pointed out that both S_2 and S_3 scale through the domain $-1 < \beta < 0$. The case of S_4 is trickier because of the absence of an explicit noise term in equation (29). We discuss it below. First we turn to extracting the multifractal behavior of S_4 and higher order structure functions. This behavior becomes relevant when the associated exponents are smaller than the scaling ones, and therefore the corresponding non-scaling term dominates over the scaling one, since $r/L \ll 1$.

We first note that in k -space both S_3 and S_4 depend on $< u(k_1)u(k_2)u(k_3) >, k_1 + k_2 + k_3 = 0$, the first one through its definition, the second one through the ϵ dependent term in (29). We thus make the following general *ansatz*

$$Im < u(k_1)u(k_2)u(k_3) > \sim \frac{|k_1|^{\mu_1}|k_2|^{\mu_2}|k_3|^{\mu_3}}{k_1 k_2 k_3} + \text{permutations} \quad (33)$$

The constraint that $S_3(r) \sim r^{-\beta}$ (cf. equation (28)) leads to $\mu_1 + \mu_2 + \mu_3 = 1 + \beta$. We can show that the lowest exponent is obtained when $\mu_1 = \mu_2 = \mu_3 = \mu/3 = (1 + \beta)/3$. Putting the *ansatz* into the 2nd term of (29) leads to

$$dS_4/dr \sim \nu \int_{-\infty}^{\infty} d\alpha dk_1 dk_2 dk_3 \sin(k_1 r) \frac{|k_1 k_2 k_3|^\mu}{k_1} \exp -i\alpha(k_1 + k_2 + k_3) \quad (34)$$

Performing the k integrals with a cutoff δ and then integrating over α , with $0 < \mu_1 < 1$, one obtains

$$dS_4/dr \sim \nu (2\pi/\delta)^{\mu_2 + \mu_3} (1/\delta) r^{-\mu_1} \quad (35)$$

and thus, with $\mu_1 = \mu_2 = \mu_3$,

$$S_4(r) \sim \frac{\nu}{\delta^{1+2\mu/3}} r^{1-\mu/3} \quad (36)$$

It is important to note here that the non-scaling behavior arises from the term in the equation which involves ϵ . The expression for S_4 contains two results:

(i) the fact that in the limit $\nu \rightarrow 0$,

$$\nu \sim \delta^{1+2(1+\beta)/3} \quad (37)$$

whereas in the scaling limit $\nu \sim \delta^{1-\beta/3}$. (By writing that at the dissipation scale δ , the characteristic eddy time $t_\delta \sim \delta/\delta^h$ is of order of the dissipation time δ^2/ν , one finds $\nu \sim \delta^{1+h}$) One thus has a new dissipation scale in S_4 , namely $\delta \sim \nu^{\frac{1}{1+h_4}}$. This dissipation scale depends on the corresponding multifractal exponent $h_4 = 2(1+\beta)/3$. For the dominant term this multifractal exponent has to be construed as the one which minimizes ζ_p (cf. (15)).

(ii) second it gives the non-scaling exponent $\zeta_4 = (2-\beta)/3$, which being smaller than the scaling exponent $\zeta_4 = -4\beta/3$ in the region $-1 < \beta < -2/3$, dominates over the scaling term in this region.

We now have to get back to the question how scaling behavior arises in S_4 . One can show that it arises through the $\nu dS_2/dr$ contribution in S_3 present in the Von Karman-Howarth relation (cf. equation (23)). It corresponds to $\mu_1 + \mu_2 + \mu_3 = 2 + 2\beta/3$ in the *ansatz* for S_3 (see above) with however $\mu_1 \neq \mu_2 = \mu_3$.

One can now proceed along the same lines to find the behavior of $S_5(r)$, taking as a starting point an *ansatz* similar to the one used for S_4 , but now for $\langle u(k_1)u(k_2)u(k_3)u(k_4) \rangle$, $k_1 + k_2 + k_3 + k_4 = 0$. There are now four μ 's, the sum of which is constrained by the known behavior of S_4 in two different regions $-1 < \beta < -2/3$ and $-2/3 < \beta < 0$. We know already that in S_5 because of the presence in equation (30) of the noise term, a scaling contribution will be present. The question that is to be settled through making the *ansatz* on the 4-point function, is whether there are regions in which the scaling term is subdominant, as happens for S_4 . The answer is yes, and one finds that there are three different regions:

- (i) $-1/2 < \beta < 0$, where scaling behavior dominates, and thus $\zeta_5 = -5\beta/3$,
- (ii) $-2/3 < \beta < -1/2$, where S_5 does not scale, $\zeta_5 = (3-4\beta)/6$, and this exponent is smaller than the scaling one and therefore the corresponding term dominates in $S_5(r)$,
- (iii) $-1 < \beta < -2/3$, where S_5 has still another multifractal exponent, $\zeta_5 = (5-\beta)/6$, which gives the dominant behavior in this region of β . The three exponents connect smoothly at the end points of each interval. In each interval all three terms are present, but the term with the smallest exponent dominates. The first four ζ_p 's are shown[6] in Figure 2.

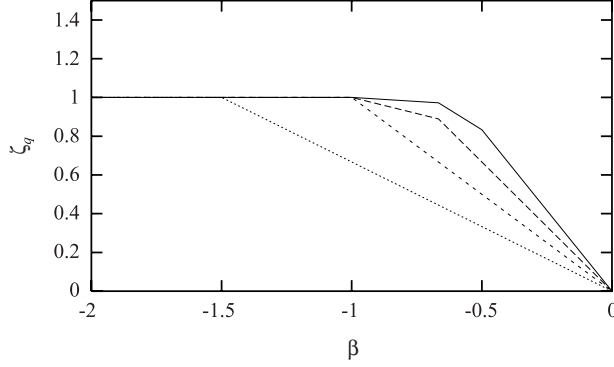


Figure 2: Exponents $\zeta_2, \zeta_3, \zeta_4$ and ζ_5 vs. β , for $-2 < \beta < 0$. The lowest curve is that of ζ_2 , the others above it are in order $\zeta_3, \zeta_4, \zeta_5$.

The following general scenario emerges from these results: as p increases, simple scaling with $\zeta_p = -p\beta/3$ occurs over a progressively diminishing range of values for β close to zero (and negative). Over most of the considered domain therefore, multiscaling occurs as soon as $p \geq 4$, with the ζ_p 's continuous and piecewise linear, the number of linear segments increasing as p gets larger. As $\beta \rightarrow -1$ all the ζ_p 's for $p \geq 3$ go towards 1. This extreme multifractal regime is a manifestation of the increasingly important role played by shocks as the noise acts on larger and larger scales.

Several remarks are in order here:

(i) if one extracts a fractal scaling exponent for velocity variations from the calculations, as we have done above for S_4 (equations (12) and (37)), one finds a different value for h_5 in each of the three regions of β , where different ζ_5 's dominate, namely $h_5 = -\beta/3$ for $-1/2 < \beta < 0$, $h_5 = 1/2 + 2\beta/3$ for $-2/3 < \beta < -1/2$, and $h_5 = (1 + \beta)/6$ for $-1 < \beta < -2/3$. Thus h_5 is continuous and piecewise linear, and goes to zero as $\beta \rightarrow -1$, which is a reflection of the increasing dominance of shocks. The same is true for all h_p 's with $p \geq 4$.

(ii) one can also calculate continuous and piecewise linear fractal dimensions $D(h_p)$ with the help of equation (15), assuming that the corresponding h_p minimizes the right hand side, and using the values of h_p and ζ_p which result from the "ansatz" calculation. One finds that all fractal dimensions tend towards zero as $\beta \rightarrow -1$, which again is consistent with the dominance of shock structure.

(ii) we cannot show in general that our calculation based on an *ansatz* in k -space, and the assumption of the equality of μ 's in S_5 (cf. equations

(33) and (34)) leads to the "true" dominant behavior in each domain. It is possible that our continuous, piecewise linear ζ_p 's, are only an approximation to the "true" function $\zeta_p(\beta)$.

III.2. Dissipation rate correlation and intermittency.

By studying the full equation satisfied by the dissipation rate correlation

$$\langle \epsilon(x+r)\epsilon(x) \rangle \sim \langle \epsilon \rangle^2 (r/L)^{-\mu} \quad (38)$$

we are able to find expressions for the intermittency exponent μ in terms of static and dynamic exponents of velocity field correlations. Here $\epsilon(x) = \nu(\partial u/\partial x)^2$ for the Burgers equation and $\epsilon(\vec{x}) = \frac{\nu}{2}(\partial_i u_j + \partial_j u_i)^2$ for the Navier-Stokes equation are the local dissipation rates. In our previous discussion, we have taken the energy dissipation rate ϵ to be a constant, and this is all that is required to obtain Kolmogorov scaling of the structure functions. In this section $\epsilon(\vec{x})$ is considered to be a fluctuating quantity which has non trivial correlations, as experiment shows. One still has $\langle \epsilon(\vec{x}) \rangle = \epsilon = \text{constant}$ because of homogeneity.

The following two relations have been proposed for the intermittency exponent μ :

$$\mu_1 = 2 - \zeta_6 \quad (39)$$

and [10, 17]

$$\mu_2 = 2\zeta_2 - \zeta_4 \quad (40)$$

The first one, the most discussed, because experimentally the value of $\zeta_6 \approx 1.8$ agrees with that of $\mu \approx 0.25$ [18], is essentially obtained by a scaling argument, which uses the dimension of ϵ , namely V^3/L , to set $\langle \epsilon(x+r)\epsilon(x) \rangle \sim S_6(r)/r^2 \sim (r/L)^{\zeta_6-2}$.

The advantage of our approach lies in the fact that relations between μ and structure function exponents ζ_p are derived directly, and simultaneously, from the equation satisfied by the dissipation rate correlation. This equation can be derived from the stochastic Burgers or the Navier-Stokes equation by considering correlations in both space r and time τ , and then passing to the $\tau \rightarrow 0$ limit. In this limit the noise term can be expressed using the Novikov-Donsker formalism [15]. One finds in this way, with $\epsilon_1 = \epsilon(x+r, t+\tau)$, $\epsilon_2 = \epsilon(x, t)$

$$\begin{aligned} \langle \epsilon_1 \epsilon_2 \rangle &= \frac{1}{4} \partial_\tau \langle (\epsilon_1 - \epsilon_2)(u_1 - u_2)^2 \rangle - \frac{1}{6} \partial_r \langle (\epsilon_1 + \epsilon_2)(u_1 - u_2)^3 \rangle \\ &\quad - \frac{1}{4} \partial_r \langle (u_1 - u_2)^2 (\epsilon_2 u_2 - \epsilon_1 u_1) \rangle + \frac{\nu}{4} \partial_r^2 \langle (\epsilon_1 + \epsilon_2)(u_1 - u_2)^2 \rangle \end{aligned}$$

$$+\frac{1}{2} < (u_1 - u_2)(\epsilon_2 f_1 - \epsilon_1 f_2) > \quad (41)$$

The 3rd term on the right-hand side ensures Galilean invariance together with the first term (the left-hand side is Galilean invariant). The viscosity dependent term, which is connected to $d^3 S_5/dr^3$ (cf. equation (30)), goes to zero for inertial r in the zero viscosity limit.

In order to show again the mathematical similarity of expressions derived from the Burgers and Navier-Stokes equations, we show the equivalent expression in three dimensions derived from equation (2):

$$\begin{aligned} < \epsilon_1 \epsilon_2 > &= \frac{1}{4} \partial_\tau < (\epsilon_1 - \epsilon_2)(\vec{u}_1 - \vec{u}_2)^2 > - \frac{1}{4} \partial_{r_j} < (\epsilon_1 + \epsilon_2)(u_{1j} - u_{2j})(\vec{u}_1 - \vec{u}_2)^2 > \\ &+ \frac{1}{4} \partial_{r_j} < (\vec{u}_1 - \vec{u}_2)^2 (\epsilon_1 u_{1j} - \epsilon_2 u_{2j}) > \\ &+ \frac{\nu}{4} \partial_{r_j}^2 < (\epsilon_1 + \epsilon_2)(\vec{u}_1 - \vec{u}_2)^2 > + \frac{\nu}{2} \partial_{r_i} \partial_{r_j} < \epsilon_1 u_{2i} u_{2j} + \epsilon_2 u_{1i} u_{1j} > \\ &- \frac{1}{2} \partial_{r_i} < (u_{1i} - u_{2i})(\epsilon_2 p_1 + \epsilon_1 p_2) > \\ &+ \frac{1}{2} < (u_{1i} - u_{2i})(\epsilon_2 f_{1i} - \epsilon_1 f_{2i}) > \end{aligned} \quad (42)$$

Apart from the pressure term and a more complicated viscosity term due to the difference in structure of the definitions of ϵ in the Burgers and Navier-Stokes case (see the beginning of this section), the two equations correspond to each other term by term, with an obvious generalization of space indices when going from one to three dimensions.

Now going back to equation (31) with $p = 6$, one sees that the expression $< (\epsilon_1 + \epsilon_2)(u_1 - u_2)^3 >$, which occurs in (41), is precisely the term in dS_6/dr which, as argued in section III.1., leads to intermittency. Therefore from (41), $< \epsilon_1 \epsilon_2 >$ (in the $\tau \rightarrow 0$ limit) contains the intermittent behavior $(r/L)^{-\mu_1}$, with

$$\mu_1 = 2 - \zeta_6 \quad (43)$$

as given in equation (39).

As to the first term on the right hand side of (41), one can show[8] that the expression $< (\epsilon_1 - \epsilon_2)(u_1 - u_2)^2 >$ appears in $\partial S_4/\partial\tau$, where it is the only one involving the dissipation rate, and therefore leads to intermittency. There is thus a contribution here to the intermittent behavior of $< \epsilon_1 \epsilon_2 >$ of exponent

$$\mu_2 = z_{4,2} - \zeta_4 \quad (44)$$

where $z_{4,2}$ characterizes the behavior of the second order partial derivative of S_4 in time, in the limit $\tau \rightarrow 0$. The origin of μ_2 is thus dynamical. If

simple scaling in time holds, then $z_{4,2} = 2z$, where $z = 1 - h$, with the value of h equal to its scaling value. z here is the dynamical exponent, not the "frozen turbulence" exponent of value 1, which characterizes the advection of small structures by large ones. Our preceding result and remarks apply as well to Navier-Stokes turbulence. In this latter case $z = 2/3$, which is numerically equal to ζ_2 (we are going to show in III.3. that this result is general and exact). Substituting ζ_2 for z (recall that in the scaling limit $z_{4,2} = 2z$) in (44) leads to the result given in equation (40), which thus appears as a static approximation to what our derivation shows to be the dynamical intermittency exponent given by equation (44).

For the Burgers equation the two intermittency exponents of equations (43) and (44) are the two main ones. For the Navier-Stokes equation we can only assert that these same two occur as well, because our discussion does not take into account the pressure term in equation (42).

III.3. Dynamic behavior.

Except for the discussion of μ_2 in the preceding section, our concern up to now has been with the equal time structure functions. We now address the problem of their dynamical behavior. We are interested in relationships between dynamic and static exponents, and also in shedding light on Taylor's frozen turbulence hypothesis in the case when there is no average flow field. In particular we wish to understand how it happens that the square root of the rms fluctuations of the velocity field replaces the average velocity when the latter is zero, thus allowing ballistic behavior with $z = 1$ (z is defined by $\tau \sim r^z$). The objects of our study are now the space and time dependent structure functions

$$S_p(r, \tau) = \langle (u_1 - u_2)^p \rangle \quad (45)$$

where $u_1 = u(x + r, t + \tau)$, $u_2 = u(x, t)$. The generalization to the three dimensional case is straightforward.

We will concentrate on S_2 . One can derive the following equation from the stochastic Burgers equation[7]

$$\partial S_2(r, \tau) / \partial \tau = \frac{1}{2} \partial T_3 / \partial r + \langle u_1 f_2 \rangle - \langle u_2 f_1 \rangle \quad (46)$$

where

$$T_3(r, \tau) = - \langle (u_1 + u_2)(u_1 - u_2)^2 \rangle \quad (47)$$

which apart from additive constants is the same as $\langle u_1^2 u_2 + u_1 u_2^2 \rangle$. The term on the left-hand side and the first term on the right-hand side form a

Galilean invariant pair. In the $\tau \rightarrow 0$ limit T_3 does not contribute because of symmetry reasons. In this limit there is a discontinuity in the noise term because $\langle u_1 f_2 \rangle$ contributes for $\tau > 0$, and $\langle u_2 f_1 \rangle$ for $\tau < 0$. One thus has, using equations (23) and (27),

$$\partial S_2(r, \tau = 0^+)/\partial \tau = (1/L^2) \sum_k D_0 |k|^\beta \cos kr = -\frac{1}{12} dS_3/dr \quad (48)$$

Assuming simple dynamic scaling for the first time derivative of S_2 (in the $\tau \rightarrow 0$ limit), with $\tau \sim r^z$, equation (48) leads to the following relation

$$\zeta_2 - z = \zeta_3 - 1 \quad (49)$$

This equation is the same as equation (21). However here it follows from an exact equation, whereas before it was obtained from a renormalisation analysis. Moreover z here is precisely defined as the exponent which characterizes the behavior of the first order partial derivative of S_2 in time in the limit $\tau \rightarrow 0$.

Since ζ_3 is known from the Von Karman-Howarth relation (equations (22) or (23)), this equation relates the temporal and spatial exponents which characterize the behavior of the 2nd order velocity structure function. Since ζ_3 has its scaling value set by the Von Karman-Howarth relation, any scaling violations in ζ_2 has to be compensated by an equal one in z . Introducing the value of ζ_3 , one thus has in the case of the Burgers equation

$$\zeta_2 - z = -\beta - 1 \quad (50)$$

and in the case of Navier-Stokes

$$\zeta_2 - z = 0 \quad (51)$$

Thus ζ_2 and z are not independent, the knowledge of one determines the other. This is the first constraint we have found for ζ_2 , for which none can be found when one limits one's investigations to static quantities only. In particular, in the Navier-Stokes case $\zeta_2 = 2/3 = z$, whereas in the Burgers case one obtains $z = 1 + \beta/3$. The latter results are consistent with the simple Kolmogorov type scaling argument which entails $z = 1 - h$.

As $\tau \rightarrow 0$ what matters is clearly this dynamical z , the one appropriate for a Galilean invariant situation. However as soon as τ departs from zero, the ballistic behavior with $z = 1$ asserts itself. We have checked this numerically for $S_2(r = 0, \tau)$ and $S_4(r = 0, \tau)$, for which, if dynamical scaling holds and for example $S_2(r, \tau) = r^{\zeta_2} g(\tau/r^z)$, time dependence is of the form $\tau^{\zeta_2/z}$,

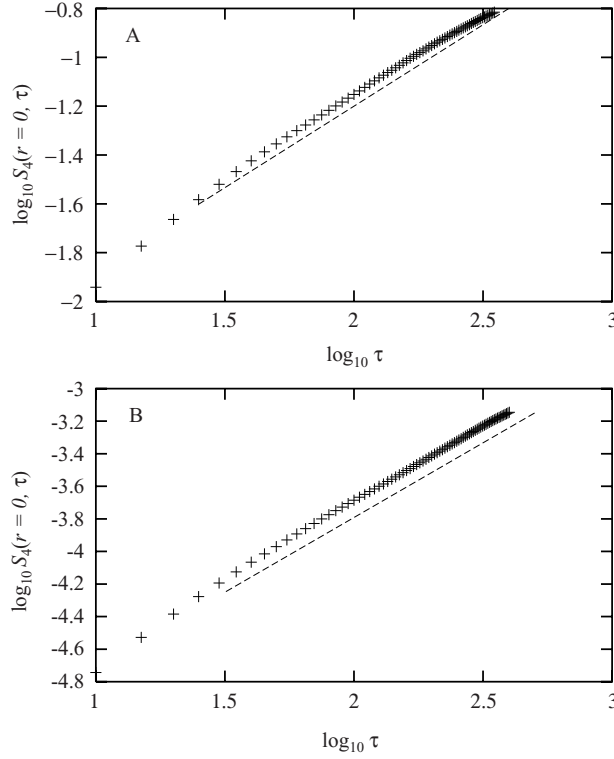


Figure 3: $\log_{10} S_4(r=0, \tau)$ vs. $\log_{10} \tau$ for a) $\beta = -0.5$ with a dashed line of slope $\zeta_4/z = 2/3$ with $z = 1$, and b) $\beta = -1$ where the dashed line has a slope of 0.92 close to the numerically observed value of ζ_4 . The expected slope is ζ_4/z , and the numerical results allows one to distinguish between $z = 1$ and $z = 2/3$, the value of $z = 1 + \beta/3$ for $\beta = -1$.

and similarly for S_4 . Numerically one is able to distinguish[7] satisfactorily between the dynamic and ballistic values of z (Figure 3). One thus verifies that as soon as τ is positive, ballistic behavior with $z = 1$ occurs.

The question now arises in which way ballistic behavior emerges, and with it the use of Taylor's frozen turbulence hypothesis, in the case when there is no average flow, i.e. $\langle u(x, t) \rangle = 0$.

In reference [7] we have shown that if one differentiates relative to τ equation (46), one is lead to the following equation

$$\partial^2 S_2(r, \tau) / \partial \tau^2 = \langle u^2 \rangle \partial^2 S_2 / \partial r^2 + \dots \quad (52)$$

The term on the right-hand side is a result of the fact that

$$\partial T_3 / \partial \tau \propto \langle u^2 \rangle \partial S_2 / \partial r \quad (53)$$

after use of the assumption that in the $\nu \rightarrow 0$ limit the term $\langle (u_1 - u_2)^2 (u_1^2 + u_2^2) \rangle \sim 2 \langle u^2 \rangle S_2$. The latter assumption arises from the observation already made by Tennekes[19] that large scale eddies advect inertial scale information past an Eulerian observer. Here we show that this assumption is encapsulated in the fact that $S_2(r, \tau)$ satisfies precisely a wave type equation of characteristic velocity given by the rms fluctuations of the velocity field. One expects this behavior to occur over time scales large compared to the dissipation time and small compared to the turnover time of the large scale structures in the system. A detailed discussion of the other terms occurring in the equation can be found in reference [7] .

IV. Remarks on intermittency.

Before embarking on these remarks one should point out that the nature of turbulence is different for the Burgers and Navier-Stokes equations: for example vortex stretching is believed to be an important ingredient in three dimensional developed turbulence.

Intermittency - the non-scaling behavior of the structure functions in the inertial range - is a hallmark of three dimensional turbulence. The language of multifractality is a convenient way to describe it. What is the origin of intermittency in the statistical behavior of turbulence? The answer is not clear, though intermittency has been connected to the presence of vortex filaments in the flow. In one experiment[20], where the size of the filaments is several times the dissipation scale, they are associated with events in the velocity field where the velocity derivative has large jumps. This is of course what happens across shocks, which play the role of coherent structures in the one dimensional stochastic Burgers equation. Here one has a clear connection between intermittency and the presence of shocks, though we are unable to give a numerical measure of the number and sizes of shocks. Typically the velocity variation across a shock occurs on length scales of the order of the dissipation scale. For β negative close to zero, shocks are barely apparent in the velocity profile, and the structure functions show scaling behavior. As β approaches -1 the shocks play a larger and larger role, and intermittency, the difference between the actual values of the ζ_p 's and their scaling values, increases correspondingly (for $p \geq 4$). At $\beta \leq -3/2$ the

shocks are present in full, dominating the velocity profile, and intermittency is extreme: all ζ_p 's are equal to 1. There is thus an obvious link between the dynamics of shocks - the small scale coherent structures - and intermittency.

We provide two other insights:

- we connect - not by a self-similarity argument, but from the exact equation - the values of the exponents measuring intermittency in the energy dissipation rate to those measuring intermittency in the velocity structure functions (see III.2.),

- we show that in the equations for the velocity structure functions the terms responsible for intermittent behavior are those which contain the energy dissipation rate. Intermittent behavior at the inertial scale is thus a consequence of dynamics which occurs at dissipation scales (see III.1.).

For the stochastic Burgers equation we are of course able to provide an extra bonus: namely, with the help of an *ansatz*, we are able to calculate from the basic equations the low order structure function exponents as β varies. Such a calculation remains the "holy grail" for statistical three dimensional turbulence.[21]

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